# Lecture Six Exercise Solutions

## Contents



### Exercise 6.1

Prove that if $P(a,b)$ factorizes, then the correlation between $a$ and $b$ is zero.

**Solution**

On page 158 we are given an expression for computing the statistical correlation between observations $\sigma_a$ and $\sigma_b$.

$$\langle \sigma_a \sigma_b \rangle - \langle \sigma_a \rangle \langle \sigma_b \rangle$$

Therefore, we need to prove that $\langle \sigma_a \sigma_b \rangle - \langle \sigma_a \rangle \langle \sigma_b \rangle = 0$ when $P(a,b)$ factorizes. We can re-write this equation as follows

$$\langle \sigma_a \sigma_b \rangle = \langle \sigma_a \rangle \langle \sigma_b \rangle \tag{1}$$

The expectation values (Section 4.7) $\langle \sigma_a \rangle$ and $\langle \sigma_b \rangle$ can be written as (Eq. 4.11)

$$\langle \sigma_a \rangle = \sum_a \sigma_a \cdot P_A(\sigma_a)$$

and

$$\langle \sigma_b \rangle = \sum_b \sigma_b \cdot P_B(\sigma_b)$$

Which are no other than weighted sums, this is, sum of possible outcomes weighted with the probability functions $P_A$ or $P_B$ respectively.

Likewise, we can write $\langle \sigma_a \sigma_b \rangle$ as

$$\langle \sigma_a \sigma_b \rangle = \sum_{a,b} \sigma_a \sigma_b \cdot P(\sigma_a, \sigma_b)$$

Therefore, equation (1) becomes

$$\sum_{a,b} \sigma_a \sigma_b \cdot P(\sigma_a, \sigma_b) = \sum_a \sigma_a \cdot P_A(\sigma_a) \sum_b \sigma_b \cdot P_B(\sigma_b)$$

If $P(\sigma_a, \sigma_b)$ factors into $P_A(\sigma_a) P_B(\sigma_b)$. The previous equation becomes

$$\sum_{a,b} \sigma_a \cdot P_A(\sigma_a) \cdot \sigma_b \cdot P_B(\sigma_b) = \sum_a \sigma_a \cdot P_A(\sigma_a) \sum_b \sigma_b \cdot P_B(\sigma_b)$$

And finally

$$\sum_a \sigma_a \cdot P_A(\sigma_a) \cdot \sum_b \sigma_b \cdot P_B(\sigma_b) = \sum_a \sigma_a \cdot P_A(\sigma_a) \sum_b \sigma_b \cdot P_B(\sigma_b)$$

Since the equality (1) is satisfied, there is no statistical correlation between observations $\sigma_a$ and $\sigma_b$ when $P(a, b)$ factorizes.

## Exercise 6.2

Show that if the two normalization conditions of Eqs. 6.4 are satisfied, then the state-vector of Eq. 6.5 is automatically normalized as well. In other words, show that for this product state, normalizing the overall state-vector does not put any additional constraints on the $\alpha$'s and $\beta$'s.

### Solution

Fore reference, Eqs. 6.4 are

$$\alpha_u^*\alpha_u + \alpha_d^*\alpha_d = 1$$

$$\beta_u^*\beta_u + \beta_d^*\beta_d = 1$$

And Eq. 5

$$|\text{product state}\rangle = \alpha_u\beta_u\,|uu\rangle + \alpha_u\beta_d\,|ud\rangle + \alpha_d\beta_u\,|du\rangle + \alpha_d\beta_d\,|dd\rangle$$

If $|\text{product state}\rangle$ is normalized, then the inner product with itself yields the unit length. This is, $\langle\text{product state}|\text{product state}\rangle = 1$.

The bra form of Eq. 5 is

$$\langle\text{product state}| = \alpha_u^*\beta_u^*\,\langle uu| + \alpha_u^*\beta_d^*\,\langle ud| + \alpha_d^*\beta_u^*\,\langle du| + \alpha_d^*\beta_d^*\,\langle dd|$$

Remembering that the inner product $\langle i|j\rangle$ between any orthonormal vectors $i$ and $j$, yields zero if $i \neq j$ and one if $i = j$. We can compute

$$\begin{aligned}\langle\text{product state}|\text{product state}\rangle = (&\alpha_u^*\beta_u^*\,\langle uu| + \alpha_u^*\beta_d^*\,\langle ud| + \alpha_d^*\beta_u^*\,\langle du| + ...\\ &\alpha_d^*\beta_d^*\,\langle dd|)\cdot(\alpha_u\beta_u\,|uu\rangle + \alpha_u\beta_d\,|ud\rangle + \alpha_d\beta_u\,|du\rangle + \alpha_d\beta_d\,|dd\rangle)\end{aligned}$$

which working out the math we get

$$\begin{aligned}\langle\text{product state}|\text{product state}\rangle = &\alpha_u^*\alpha_u\beta_u^*\beta_u + \alpha_u^*\alpha_u\beta_d^*\beta_d + ...\\ &\alpha_d^*\alpha_d\beta_u^*\beta_u + \alpha_d^*\alpha_d\beta_d^*\beta_d\end{aligned}$$

if we factor the terms we find

$$\langle\text{product state}|\text{product state}\rangle = (\alpha_u^*\alpha_u + \alpha_d^*\alpha_d)(\beta_u^*\beta_u + \beta_d^*\beta_d)$$

Finally, introducing Eqs. 6.4 we find

$$\langle\text{product state}|\text{product state}\rangle = (1)(1) = 1$$

Therefore, normalizing the product state vector does not introduce any additional constraints on the $\alpha$'s and $\beta$'s.

### Exercise 6.3

Prove that the state $|sing\rangle$ cannot be written as a product state.

**Solution**

Equation 6.5 and Section 6.6 tell us that such a product state can be described with four complex numbers, namely $\alpha_u$ and $\alpha_d$ for system $A$ and $\beta_u$ and $\beta_d$ for system $B$. With this idea in mind we will prove that $|sing\rangle$ cannot be written as a product state.

First, lets assume that we can write $|sing\rangle$ as a product state. Under that assumption, we will attempt to compute the four complex numbers (the $\alpha$'s and $\beta$'s) using the following system of equations.

$$\alpha_u\beta_u = 0$$

$$\alpha_u\beta_d = 1/\sqrt{2}$$

$$\alpha_d\beta_u = -1/\sqrt{2}$$

$$\alpha_d\beta_u = 0$$

However, it is easy to realize that we can not satisfy the previous system of equations (e.g. first and second suggest that $\beta_u = 0$ but our third equation suggests otherwise) and therefore $|sing\rangle$ cannot be written as a product state.

### Exercise 6.4

Use the matrix forms of $\sigma_z$,$\sigma_x$, and $\sigma_y$ and the column vectors for $|u\}$ and $|d\}$ to verify Eqs. 6.6. Then, use Eqs. 6.6 and 6.7 to write the equations that were left out of Eqs. 6.8. Use the appendix to check your answers.

**Solution**

First we need to verify (Equations 6.6) that

$$\sigma_z|u\} = |u\}$$

$$\sigma_z|d\} = -|d\}$$

$$\sigma_x|u\} = |d\}$$

$$\sigma_x|d\} = |u\}$$

$$\sigma_y|u\} = i|d\}$$

$$\sigma_y|u\} = -i|u\}$$

The spin operators $\sigma_z$,$\sigma_x$, and $\sigma_y$ in matrix form are

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

And the column vectors for $|u\}$ and $|d\}$ are

$$|u\} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

$$|d\} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

Now we apply each operator to each vector and verify with Eqs. 6.6

$$\sigma_z|u\} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = |u\}$$

$$\sigma_z|d\} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1 \end{Bmatrix} = -|d\}$$

$$\sigma_x|u\} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = |d\}$$

$$\sigma_x|d\} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = |u\}$$

$$\sigma_y|u\} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ i \end{Bmatrix} = i|d\}$$

$$\sigma_y|u\} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} -i \\ 0 \end{Bmatrix} = -i|u\}$$

The expressions above are consistent with Eqs. 6.6 and therefore we verified them using matrix forms of spin operators and column vectors. Now we need to use Eqs. 6.6 and 6.7 to write the equations that were left out of Eqs. 6.8. Equations 6.7 are

$$\tau_z|u\} = |u\}$$

$$\tau_z|d\} = -|d\}$$

$$\tau_x|u\} = |d\}$$

$$\tau_x|d\} = |u\}$$

$$\tau_y|u\} = i|d\}$$

$$\tau_y|u\} = -i|u\}$$

Also, Eqs. 6.8 are

$$\sigma_z|uu\} = |u\}$$

$$\sigma_z|du\} = -|d\}$$

$$\sigma_x|ud\} = |d\}$$

$$\sigma_x|dd\} = |u\}$$

$$\sigma_y|uu\} = i|d\}$$

$$\sigma_y|du\} = -i|u\}$$

$$\tau_z|uu\} = |u\}$$

$$\tau_z|du\} = -|d\}$$

$$\tau_x|ud\} = |d\}$$

$$\tau_x|du\} = |u\}$$

$$\tau_y|uu\} = i|d\}$$

$$\tau_y|dd\} = -i|u\}$$

The equations that were left out can be viewed as blank entries in the next table

| | $\boldsymbol{|uu\rangle}$ | $\boldsymbol{|ud\rangle}$ | $\boldsymbol{|du\rangle}$ | $\boldsymbol{|dd\rangle}$ |
|---|---|---|---|---|
| $\boldsymbol{\sigma_z}$ | $|uu\rangle$ | | $-|du\rangle$ | |
| $\boldsymbol{\sigma_x}$ | | $|dd\rangle$ | | $|ud\rangle$ |
| $\boldsymbol{\sigma_y}$ | $i\,|du\rangle$ | | $-i\,|uu\rangle$ | |
| $\boldsymbol{\tau_z}$ | $|uu\rangle$ | | $|du\rangle$ | |
| $\boldsymbol{\tau_x}$ | | $|uu\rangle$ | $|dd\rangle$ | |
| $\boldsymbol{\tau_y}$ | $i\,|ud\rangle$ | | | $-i\,|du\rangle$ |

In order to fill out the rest of the table we will proceed as follows for the entry in the first row and second column

$$\sigma_z |ud\rangle = (\sigma_z \otimes I) \cdot (|u\} \otimes |d\rangle)$$

$$= (\sigma_z|u\}) \otimes (I\,|d\rangle)$$

$$= (|u\}) \otimes (|d\rangle)$$

$$= |ud\rangle$$

Replicating this procedure for the rest of the blank entries we can complete the table as follows

| | $\boldsymbol{|uu\rangle}$ | $\boldsymbol{|ud\rangle}$ | $\boldsymbol{|du\rangle}$ | $\boldsymbol{|dd\rangle}$ |
|---|---|---|---|---|
| $\boldsymbol{\sigma_z}$ | $|uu\rangle$ | $|ud\rangle$ | $-\,|du\rangle$ | $-\,|dd\rangle$ |
| $\boldsymbol{\sigma_x}$ | $|du\rangle$ | $|dd\rangle$ | $|uu\rangle$ | $|ud\rangle$ |
| $\boldsymbol{\sigma_y}$ | $i\,|du\rangle$ | $i\,|dd\rangle$ | $-i\,|uu\rangle$ | $-i\,|ud\rangle$ |
| $\boldsymbol{\tau_z}$ | $|uu\rangle$ | $-\,|ud\rangle$ | $|du\rangle$ | $-\,|dd\rangle$ |
| $\boldsymbol{\tau_x}$ | $|ud\rangle$ | $|uu\rangle$ | $|dd\rangle$ | $|du\rangle$ |
| $\boldsymbol{\tau_y}$ | $i\,|ud\rangle$ | $-i\,|uu\rangle$ | $i\,|dd\rangle$ | $-i\,|du\rangle$ |

## Exercise 6.5

Prove the following theorem:

When any of Alice's or Bob's spin operators acts on a product state, the result is still a product state.

Show that in a product state, the expectation value of any component of $\vec{\sigma}$ or $\vec{\tau}$ is exactly the same as it would be in the individual single spin-states.

**Solution**

First we show that when any of Allice's or Bob's spin operators acts on a product state, the result is still a product state.

A product state can be defined as follows (Eq. 6.5)

$$|\text{product state}\rangle = \{\alpha_u|u\} \ + \alpha_d|d\}\} \otimes \{\beta_u\,|u\rangle + \beta_d\,|d|\rangle\}$$

Where the $\alpha$ and $\beta$ terms are constants. Now if any of Alice's ($\sigma_n$) or Bob's ($\tau_n$) spin operators act on the product state we find

$$\{\sigma_n \otimes \tau_n\} \cdot |\text{product state}\rangle = \{\sigma_n \cdot (\alpha_u|u\} \ + \alpha_d|d\})\} \otimes \{\tau_n \cdot (\beta_u|u\} \ + \beta_d|d\})\}$$

$$= \{\gamma_u|u\} \ + \gamma_d|d\}\} \otimes \{\delta_u\,|u\rangle + \delta_d\,|d|\rangle\}$$

Where the $\gamma$ and $\delta$ terms are constants. The previous expression continues to satisfy the definition of a product state.

For the later part we need to show that the expectation value of any component of $\vec{\sigma}$ or $\vec{\tau}$ is the same the same in both product state and single-spin states.

The most general form of a component of $\vec{\sigma}$ is given in Eq. 3.22 (and Eq. 3.23) as follows

$$\begin{aligned}\sigma_n &= \vec{\sigma} \cdot \vec{n} \\ &= \sigma_x n_x + \sigma_y n_y + \sigma_z n_z \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} n_x + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} n_y + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} n_z \\ &= \begin{pmatrix} n_z & (n_x - i n_y) \\ (n_x + i n_y) & -n_z \end{pmatrix}\end{aligned}$$

Therefore, we can also define a component of $\vec{\tau}$ as

$$\tau_m = \begin{pmatrix} m_z & (m_x - i m_y) \\ (m_x + i m_y) & -m_z \end{pmatrix}$$

The bra and ket vectors of Alice's single spin state (Beginning of Subsection 6.5 p.163) are

$$|A\rangle = \alpha_u|u\} \ + \alpha_d|d\}$$

$$\langle A| = \alpha_u^*\{u| \; + \alpha_d^*\{d|$$

Now the expectation value of the component $\sigma_n$ is given by

$$\begin{aligned}
\langle A|\,\sigma_n\,|A\rangle &= \langle A| \begin{pmatrix} n_z & (n_x - in_y) \\ (n_x + in_y) & -n_z \end{pmatrix} (\alpha_u \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} + \alpha_d \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}) \\
&= \langle A| \left( \alpha_u \begin{Bmatrix} n_z \\ (n_x + in_y) \end{Bmatrix} + \alpha_d \begin{Bmatrix} n_x - in_y \\ -n_z \end{Bmatrix} \right) \\
&= (\alpha_u^*\{u| \; + \alpha_d^*\{d|) \left( \begin{Bmatrix} \alpha_u n_z + \alpha_d(n_x - in_y) \\ \alpha_u(n_x + in_y) - \alpha_d n_z \end{Bmatrix} \right) \\
&= \left( \alpha_u^* \begin{Bmatrix} 1 \; 0 \end{Bmatrix} + \alpha_d^* \begin{Bmatrix} 0 \; 1 \end{Bmatrix} \right) \left( \begin{Bmatrix} \alpha_u n_z + \alpha_d(n_x - in_y) \\ \alpha_u(n_x + in_y) - \alpha_d n_z \end{Bmatrix} \right) \\
&= \alpha_u^*\{\alpha_u n_z + \alpha_d(n_x - in_y)\} + \alpha_d^*\{\alpha_u(n_x + in_y) - \alpha_d n_z\}
\end{aligned}$$

Rearranging the previous expression we find

$$\langle A|\,\sigma_n\,|A\rangle = n_x(\alpha_u^*\alpha_d + \alpha_d^*\alpha_u) - in_y(\alpha_u^*\alpha_d - \alpha_d^*\alpha_u) + n_z(\alpha_u^*\alpha_u - \alpha_d^*\alpha_d)$$

In a similar fashion we find that the expected value for the component $\tau_n$ in Bob's single spin state is

$$\langle B|\,\tau_m\,|B\rangle = m_x(\beta_u^*\beta_d + \beta_d^*\beta_u) - im_y(\beta_u^*\beta_d - \beta_d^*\beta_u) + m_z(\beta_u^*\beta_u - \beta_d^*\beta_d)$$

Now we need to compute expectation values of $\sigma_n$ and $\tau_n$ in the product state. To keep the computations tractable yet explicit we will use the components in the form

$$\sigma_n = (\sigma_x n_x + \sigma_y n_y + \sigma_z n_z) \otimes I$$

$$\tau_m = I \otimes (\tau_x m_x + \tau_y m_y + \tau_z m_z)$$

instead of the matrix and column vector form. We define a product state $|PS\rangle$

$$|PS\rangle = \{\alpha_u |u\} \ + \alpha_d |d\}\} \otimes \{\beta_u |u\rangle + \beta_d |d\rangle\}$$

Or in bra form

$$\langle PS| = \{\alpha_u^* \{u| + \alpha_d^* \{d|\} \otimes \{\beta_u^* \langle u| + \beta_d^* \langle d|\}$$

First we compute the expected value of $\sigma_n$ in $|PS\rangle$

$$\begin{aligned}
\langle PS| \sigma_n |PS\rangle &= \langle PS| [\{(\sigma_x n_x + \sigma_y n_y + \sigma_z n_z) \otimes I\}\{(\alpha_u |u\} \ + \alpha_d |d\}) \otimes (\beta_u |u\rangle + \beta_d |d\rangle)\}] \\
&= \langle PS| [\{(\sigma_x n_x + \sigma_y n_y + \sigma_z n_z)(\alpha_u |u\} \ + \alpha_d |d\})\} \otimes \{I(\beta_u |u\rangle + \beta_d |d\rangle)\}] \\
&= \langle PS| [\{((n_x + i n_y)\alpha_u - n_z \alpha_d)|d\} + ((n_x - i n_y)\alpha_d + n_z \alpha_u)|u\}\} \otimes \{\beta_u |u\rangle + \beta_d |d\rangle\}] \\
&= [\{\alpha_u^* \{u| + \alpha_d^* \{d|\} \otimes \{\beta_u^* \langle u| + \beta_d^* \langle d|\}] \ \cdot [\{((n_x + i n_y)\alpha_u - n_z \alpha_d)|d\} + ... \\
&\qquad ((n_x - i n_y)\alpha_d + n_z \alpha_u)|u\}\} \otimes \{\beta_u |u\rangle + \beta_d |d\rangle\}] \\
&= \{\alpha_u^* \{u| + \alpha_d^* \{d|\}\{((n_x + i n_y)\alpha_u - n_z \alpha_d)|d\} + ((n_x - i n_y)\alpha_d + n_z \alpha_u)|u\}\} \otimes ... \\
&\qquad \{\beta_u^* \langle u| + \beta_d^* \langle d|\}\{\beta_u |u\rangle + \beta_d |d\rangle\} \\
&= \{\alpha_u^*((n_x - i n_y)\alpha_d + n_z \alpha_u) + \alpha_d^*((n_x + i n_y)\alpha_u - n_z \alpha_d)\} \otimes \{\beta_u^* \beta_u + \beta_d^* \beta_d\}
\end{aligned}$$

Rearranging we find

$$\langle PS| \sigma_n |PS\rangle = \{n_x(\alpha_u^* \alpha_d + \alpha_d^* \alpha_u) - i n_y(\alpha_u^* \alpha_d - \alpha_d^* \alpha_u) + n_z(\alpha_u^* \alpha_u - \alpha_d^* \alpha_d)\} \otimes \{\beta_u^* \beta_u + \beta_d^* \beta_d\}$$

Likewise, we can compute the expected value of $\tau_m$ in $|PS\rangle$ and we find

$$\langle PS|\,\tau_m\,|PS\rangle = \{\alpha_u^*\alpha_u+\alpha_d^*\alpha_d\}\otimes\{m_x(\beta_u^*\beta_d+\beta_d^*\beta_u)-im_y(\beta_u^*\beta_d-\beta_d^*\beta_u)+m_z(\beta_u^*\beta_u-\beta_d^*\beta_d)\}$$

It is easy to realize that imposing the normalization conditions in Eqs. 6.4

$$\alpha_u^*\alpha_u + \alpha_d^*\alpha_d = 1$$

$$\beta_u^*\beta_u + \beta_d^*\beta_d = 1$$

the previous expressions reduce the single-spin cases. This is

$$\begin{aligned}\langle PS|\,\sigma_n\,|PS\rangle &= \{n_x(\alpha_u^*\alpha_d + \alpha_d^*\alpha_u) - in_y(\alpha_u^*\alpha_d - \alpha_d^*\alpha_u) + n_z(\alpha_u^*\alpha_u - \alpha_d^*\alpha_d)\} \otimes 1 \\ &= n_x(\alpha_u^*\alpha_d + \alpha_d^*\alpha_u) - in_y(\alpha_u^*\alpha_d - \alpha_d^*\alpha_u) + n_z(\alpha_u^*\alpha_u - \alpha_d^*\alpha_d)\end{aligned}$$

and

$$\begin{aligned}\langle PS|\,\tau_m\,|PS\rangle &= 1 \otimes \{m_x(\beta_u^*\beta_d + \beta_d^*\beta_u) - im_y(\beta_u^*\beta_d - \beta_d^*\beta_u) + m_z(\beta_u^*\beta_u - \beta_d^*\beta_d)\} \\ &= m_x(\beta_u^*\beta_d + \beta_d^*\beta_u) - im_y(\beta_u^*\beta_d - \beta_d^*\beta_u) + m_z(\beta_u^*\beta_u - \beta_d^*\beta_d)\end{aligned}$$

Therefore we proved that

$$\langle PS|\,\sigma_n\,|PS\rangle = \langle A|\,\sigma_n\,|A\rangle$$

$$\langle PS|\,\tau_m\,|PS\rangle = \langle B|\,\tau_m\,|B\rangle$$

## Exercise 6.6

Assume Charlie has prepared the two spins in the singlet state. This time, Bob measures $\tau_y$ and Alice measures $\sigma_x$. What is the expectation value of $\sigma_x \tau_y$?

What does this say about the correlation between the two measurements?

**Solution**

Using the table of page 350 (appendix of the book) we proceed computing the expectation value of $\sigma_x \tau_y$ as follows

$$\begin{aligned}
\langle sing| \sigma_x \tau_y |sing\rangle &= \langle sing| \sigma_x \tau_y \tfrac{1}{\sqrt{2}}(|ud\rangle - |du\rangle) \\
&= \langle sing| \sigma_x \tfrac{1}{\sqrt{2}}(-i\,|uu\rangle - i\,|dd\rangle) \\
&= \langle sing| \tfrac{1}{\sqrt{2}}(-i\,|du\rangle - i\,|ud\rangle) \\
&= \tfrac{1}{\sqrt{2}}(\langle ud| - \langle du|)\tfrac{1}{\sqrt{2}}(-i\,|du\rangle - i\,|ud\rangle) \\
&= \tfrac{1}{2}(0 - i + 0 + i) \\
&= 0
\end{aligned}$$

*I think* that there is no correlation between the measurements because the expectation value of $\sigma_x \tau_y$ is 0, this is, the outcome of the composite observable is completely uncertain even though we know the state-vector (as in we know the system but anything about its parts).

## Exercise 6.7

Next Charlie prepares the spins in a different state, called $|T_1\rangle$ , where

$$|T_1\rangle = \frac{1}{\sqrt{2}}(|ud\rangle + |du\rangle)$$

In these examples, $T$ stands for *triple*. These triplet states are completely different from the states in the coin and die examples. What are the expectation values of the operators $\sigma_z \tau_z$, $\sigma_x \tau_x$, and $\sigma_y \tau_y$?

What a difference a sign can make!

**Solution**

We compute the expectation values for the operators $\sigma_z\tau_z$, $\sigma_x\tau_x$, and $\sigma_y\tau_y$; using again the table in page 350 as follows

$$
\begin{aligned}
\langle T_1|\,\sigma_z\tau_z\,|T_1\rangle &= \langle T_1|\,\sigma_z\tau_z\tfrac{1}{\sqrt{2}}(|ud\rangle + |du\rangle) \\
&= \langle T_1|\,\sigma_z\tfrac{1}{\sqrt{2}}(-\,|ud\rangle + |du\rangle) \\
&= \langle T_1|\,\tfrac{1}{\sqrt{2}}(-\,|ud\rangle - |du\rangle) \\
&= \tfrac{1}{\sqrt{2}}(\langle ud| + \langle du|)\tfrac{1}{\sqrt{2}}(-\,|ud\rangle - |du\rangle) \\
&= \tfrac{1}{2}(-1-0-0-1) \\
&= -1
\end{aligned}
$$

$$
\begin{aligned}
\langle T_1|\,\sigma_x\tau_x\,|T_1\rangle &= \langle T_1|\,\sigma_x\tau_x\tfrac{1}{\sqrt{2}}(|ud\rangle + |du\rangle) \\
&= \langle T_1|\,\sigma_x\tfrac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle) \\
&= \langle T_1|\,\tfrac{1}{\sqrt{2}}(|du\rangle + |ud\rangle) \\
&= \tfrac{1}{\sqrt{2}}(\langle ud| + \langle du|)\tfrac{1}{\sqrt{2}}(|du\rangle + |ud\rangle) \\
&= \tfrac{1}{2}(0+1+1+0) \\
&= +1
\end{aligned}
$$

$$
\begin{aligned}
\langle T_1|\,\sigma_y\tau_y\,|T_1\rangle &= \langle T_1|\,\sigma_y\tau_y\tfrac{1}{\sqrt{2}}(|ud\rangle + |du\rangle) \\
&= \langle T_1|\,\sigma_y\tfrac{1}{\sqrt{2}}(i\,|uu\rangle - i\,|dd\rangle) \\
&= \langle T_1|\,\tfrac{1}{\sqrt{2}}(-i^2\,|du\rangle - i^2\,|ud\rangle) \\
&= \tfrac{1}{\sqrt{2}}(\langle ud| + \langle du|)\tfrac{1}{\sqrt{2}}(|du\rangle + |ud\rangle) \\
&= \tfrac{1}{2}(0+1+1+0) \\
&= +1
\end{aligned}
$$

## Exercise 6.8

Do the same for the other two entangled triplet states,

$$
|T_2\rangle = \frac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle)
$$

$$
|T_3\rangle = \frac{1}{\sqrt{2}}(|uu\rangle - |dd\rangle)
$$

### Solution

We start with $|T_2\rangle$ as follows

$$
\begin{aligned}
\langle T_2|\,\sigma_z\tau_z\,|T_2\rangle &= \langle T_2|\,\sigma_z\tau_z\tfrac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle) \\
&= \langle T_2|\,\sigma_z\tfrac{1}{\sqrt{2}}(|uu\rangle - |dd\rangle) \\
&= \langle T_2|\,\tfrac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle) \\
&= \tfrac{1}{\sqrt{2}}(\langle uu| + \langle dd|)\tfrac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle) \\
&= \tfrac{1}{2}(1+0+0+1) \\
&= +1
\end{aligned}
$$

$$\begin{aligned}
\langle T_2|\,\sigma_x\tau_x\,|T_2\rangle &= \langle T_2|\,\sigma_x\tau_x\tfrac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle) \\
&= \langle T_2|\,\sigma_x\tfrac{1}{\sqrt{2}}(|ud\rangle + |du\rangle) \\
&= \langle T_2|\,\tfrac{1}{\sqrt{2}}(|dd\rangle + |uu\rangle) \\
&= \tfrac{1}{\sqrt{2}}(\langle uu| + \langle dd|)\tfrac{1}{\sqrt{2}}(|dd\rangle + |uu\rangle) \\
&= \tfrac{1}{2}(0+1+1+0) \\
&= +1
\end{aligned}$$

$$\begin{aligned}
\langle T_2|\,\sigma_y\tau_y\,|T_2\rangle &= \langle T_2|\,\sigma_y\tau_y\tfrac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle) \\
&= \langle T_2|\,\sigma_y\tfrac{1}{\sqrt{2}}(i\,|ud\rangle - i\,|du\rangle) \\
&= \langle T_2|\,\tfrac{1}{\sqrt{2}}(i^2\,|dd\rangle + i^2\,|uu\rangle) \\
&= \tfrac{1}{\sqrt{2}}(\langle uu| + \langle dd|)\tfrac{1}{\sqrt{2}}(-\,|dd\rangle - |uu\rangle) \\
&= \tfrac{1}{2}(-0-1-1-0) \\
&= -1
\end{aligned}$$

Now we do the same for $|T_3\rangle$

$$\begin{aligned}
\langle T_3|\,\sigma_z\tau_z\,|T_3\rangle &= \langle T_3|\,\sigma_z\tau_z\tfrac{1}{\sqrt{2}}(|uu\rangle - |dd\rangle) \\
&= \langle T_3|\,\sigma_z\tfrac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle) \\
&= \langle T_3|\,\tfrac{1}{\sqrt{2}}(|uu\rangle - |dd\rangle) \\
&= \tfrac{1}{\sqrt{2}}(\langle uu| - \langle dd|)\tfrac{1}{\sqrt{2}}(|uu\rangle - |dd\rangle) \\
&= \tfrac{1}{2}(1-0-0+1) \\
&= +1
\end{aligned}$$

$$
\begin{aligned}
\langle T_3|\,\sigma_x\tau_x\,|T_3\rangle &= \langle T_3|\,\sigma_x\tau_x\tfrac{1}{\sqrt{2}}(|uu\rangle - |dd\rangle) \\
&= \langle T_3|\,\sigma_x\tfrac{1}{\sqrt{2}}(|ud\rangle - |du\rangle) \\
&= \langle T_3|\,\tfrac{1}{\sqrt{2}}(|dd\rangle - |uu\rangle) \\
&= \tfrac{1}{\sqrt{2}}(\langle uu| - \langle dd|)\tfrac{1}{\sqrt{2}}(|dd\rangle - |uu\rangle) \\
&= \tfrac{1}{2}(0 - 1 - 1 + 0) \\
&= -1
\end{aligned}
$$

$$
\begin{aligned}
\langle T_3|\,\sigma_y\tau_y\,|T_3\rangle &= \langle T_3|\,\sigma_y\tau_y\tfrac{1}{\sqrt{2}}(|uu\rangle - |dd\rangle) \\
&= \langle T_3|\,\sigma_y\tfrac{1}{\sqrt{2}}(i\,|ud\rangle + i\,|du\rangle) \\
&= \langle T_3|\,\tfrac{1}{\sqrt{2}}(i^2\,|dd\rangle - i^2\,|uu\rangle) \\
&= \tfrac{1}{\sqrt{2}}(\langle uu| - \langle dd|)\tfrac{1}{\sqrt{2}}(-\,|dd\rangle + |uu\rangle) \\
&= \tfrac{1}{2}(-0 + 1 + 1 - 0) \\
&= +1
\end{aligned}
$$

In the end, we can summarize the results for Exercises 6.7 and 6.8 as follows

| | $\boldsymbol{\lvert T_1\rangle = \frac{1}{\sqrt{2}}(\lvert ud\rangle + \lvert du\rangle)}$ | $\boldsymbol{\lvert T_2\rangle = \frac{1}{\sqrt{2}}(\lvert uu\rangle + \lvert dd\rangle)}$ | $\boldsymbol{\lvert T_3\rangle = \frac{1}{\sqrt{2}}(\lvert uu\rangle - \lvert dd\rangle)}$ |
|---|---|---|---|
| $\boldsymbol{\sigma_x\tau_x}$ | $+1$ | $+1$ | $-1$ |
| $\boldsymbol{\sigma_y\tau_y}$ | $+1$ | $-1$ | $+1$ |
| $\boldsymbol{\sigma_z\tau_z}$ | $-1$ | $+1$ | $+1$ |

We can repeat the same analysis for the $|sing\rangle$ state as it will be useful for the next question. Repeating the previous steps that we did for $|T_1\rangle$, $|T_2\rangle$, and $|T_3\rangle$ we get

| | $\boldsymbol{\lvert sing\rangle = \frac{1}{\sqrt{2}}(\lvert ud\rangle - \lvert du\rangle)}$ |
|---|---|
| $\boldsymbol{\sigma_x\tau_x}$ | -1 |
| $\boldsymbol{\sigma_y\tau_y}$ | -1 |
| $\boldsymbol{\sigma_z\tau_z}$ | -1 |

## Exercise 6.9

Prove that the four vectors $|sing\rangle$, $|T_1\rangle$, $|T_2\rangle$, and $|T_3\rangle$ are eigenvectors of $\vec{\sigma} \cdot \vec{\tau}$. What are their eigenvalues?

### Solution

IF $|\lambda\rangle$ is an eigenvector of $\mathbf{M}$ then

$$\mathbf{M}\,|\lambda\rangle = \lambda\,|\lambda\rangle$$

where the constant $\lambda$ is eigenvalue. With this in mind we need to find the same structure for the operator $\vec{\sigma} \cdot \vec{\tau}$ acting on the state vectors $|sing\rangle$, $|T_1\rangle$, $|T_2\rangle$, and $|T_3\rangle$. Let's begin with $|sing\rangle$

$$\begin{aligned}
\vec{\sigma} \cdot \vec{\tau}\,|sing\rangle &= (\sigma_x\tau_x + \sigma_y\tau_y + \sigma_x\tau_x)\,|sing\rangle \\
&= \sigma_x\tau_x\,|sing\rangle + \sigma_y\tau_y\,|sing\rangle + \sigma_z\tau_z\,|sing\rangle
\end{aligned}$$

Using the tables that we derived for the previous exercise for $|sing\rangle$ we get

$$\begin{aligned}
\vec{\sigma} \cdot \vec{\tau}\,|sing\rangle &= (-1\,|sing\rangle) + (-1\,|sing\rangle) + (-1\,|sing\rangle) \\
&= -3\,|sing\rangle
\end{aligned}$$

Doing the same procedure for $|T_1\rangle$ we get

$$\begin{aligned}
\vec{\sigma} \cdot \vec{\tau}\,|T_1\rangle &= (\sigma_x\tau_x + \sigma_y\tau_y + \sigma_x\tau_x)\,|T_1\rangle \\
&= \sigma_x\tau_x\,|T_1\rangle + \sigma_y\tau_y\,|T_1\rangle + \sigma_z\tau_z\,|T_1\rangle \\
&= 1\,|T_1\rangle + 1\,|T_1\rangle - 1\,|T_1\rangle \\
&= +1\,|T_1\rangle
\end{aligned}$$

Likewise

$$\begin{aligned}
\vec{\sigma} \cdot \vec{\tau}\,|T_2\rangle &= (\sigma_x\tau_x + \sigma_y\tau_y + \sigma_x\tau_x)\,|T_2\rangle \\
&= \sigma_x\tau_x\,|T_2\rangle + \sigma_y\tau_y\,|T_2\rangle + \sigma_z\tau_z\,|T_2\rangle \\
&= 1\,|T_2\rangle - 1\,|T_2\rangle + 1\,|T_2\rangle \\
&= +1\,|T_1\rangle
\end{aligned}$$

And finally

$$\begin{aligned}
\vec{\sigma}\cdot\vec{\tau}\,|T_3\rangle &= (\sigma_x\tau_x + \sigma_y\tau_y + \sigma_x\tau_x)\,|T_3\rangle \\
&= \sigma_x\tau_x\,|T_3\rangle + \sigma_y\tau_y\,|T_3\rangle + \sigma_z\tau_z\,|T_3\rangle \\
&= -1\,|T_3\rangle + 1\,|T_3\rangle + 1\,|T_3\rangle \\
&= +1\,|T_3\rangle
\end{aligned}$$

## Exercise 6.10

A system of two spins has the Hamiltonian

$$\mathbf{H} = \frac{\omega}{2}\,\vec{\sigma}\cdot\vec{\tau}$$

What are the possible energies of the system, and what are the eigenvectors of the Hamiltonian?

Suppose the system starts in the state $|uu\rangle$. What is the state at any later time? answer the same question for initial states of $|ud\rangle$, $|du\rangle$, and $|dd\rangle$.

**Solution**

In the previous exercise, we computed the eigenvectors of $\vec{\sigma}\cdot\vec{\tau}$ and found an eigenvalue for each of them . Considering the constant $\frac{\omega}{2}$ multiplying the observable $\vec{\sigma}\cdot\vec{\tau}$, we find that the eigenvalues are

$$\lambda_{sing} = -\tfrac{3\omega}{2} \quad , \quad \lambda_{T_1} = \tfrac{\omega}{2} \quad , \quad \lambda_{T_2} = \tfrac{\omega}{2} \quad , \quad \lambda_{T_3} = \tfrac{\omega}{2}$$

If the possible energies are captured in the observable $\mathbf{H}$ an are 'something we can measure' with certain probability, then the eigenvalues of $\mathbf{H}$ represent the possible energies.

For the later part, since we have computed the eigenvectors and know the that the initial state vector is $|uu\rangle$, we have all the "ingredients" to proceed with the recipe of subsection 4.13 (pp 124) to study time-dependence

Recipe for a Schröndiger ket:

1. We already have the Hamiltonian operator **H**

$$\mathbf{H} = \frac{\omega}{2}\,\vec{\sigma}\cdot\vec{\tau}$$

2. The initial state is

$$|\Psi(0)\rangle = |uu\rangle$$

3. The eigenvalues and eigenvectors are

$$\mathbf{H}\,|sing\rangle = -\frac{3\omega}{2}\,|sing\rangle$$
$$\mathbf{H}\,|T_1\rangle = \frac{\omega}{2}\,|T_1\rangle$$
$$\mathbf{H}\,|T_2\rangle = \frac{\omega}{2}\,|T_2\rangle$$
$$\mathbf{H}\,|T_3\rangle = \frac{\omega}{2}\,|T_3\rangle$$

4. The coefficients $\alpha_j(0)$ are

$$\begin{aligned}\alpha_{sing}(0) &= \langle sing|\,|\Psi(0)\rangle \\ &= \tfrac{1}{\sqrt{2}}(\langle ud| - \langle du|)(|uu\rangle) \\ &= 0\end{aligned}$$

$$\begin{aligned}\alpha_{T_1}(0) &= \langle T_1|\,|\Psi(0)\rangle \\ &= \tfrac{1}{\sqrt{2}}(\langle ud| + \langle du|)(|uu\rangle) \\ &= 0\end{aligned}$$

$$\begin{aligned}\alpha_{T_2}(0) &= \langle T_2|\,|\Psi(0)\rangle \\ &= \tfrac{1}{\sqrt{2}}(\langle uu| + \langle dd|)(|uu\rangle) \\ &= \tfrac{1}{\sqrt{2}}\end{aligned}$$

$$\begin{aligned}\alpha_{T_3}(0) &= \langle T_3|\,|\Psi(0)\rangle \\ &= \tfrac{1}{\sqrt{2}}(\langle uu| - \langle dd|)(|uu\rangle) \\ &= \tfrac{1}{\sqrt{2}}\end{aligned}$$

5. Now we write the inital state vector in terms of the eigenvectors and $\alpha_j(0)$ coefficients we just computed

$$|\Psi(0)\rangle = \alpha_{T_2}(0)\,|T_2\rangle + \alpha_{T_3}(0)\,|T_3\rangle$$

6. Now we introduce the time-dependence coefficients

$$|\Psi(t)\rangle = \alpha_{T_2}(t)\,|T_2\rangle + \alpha_{T_3}(t)\,|T_3\rangle$$

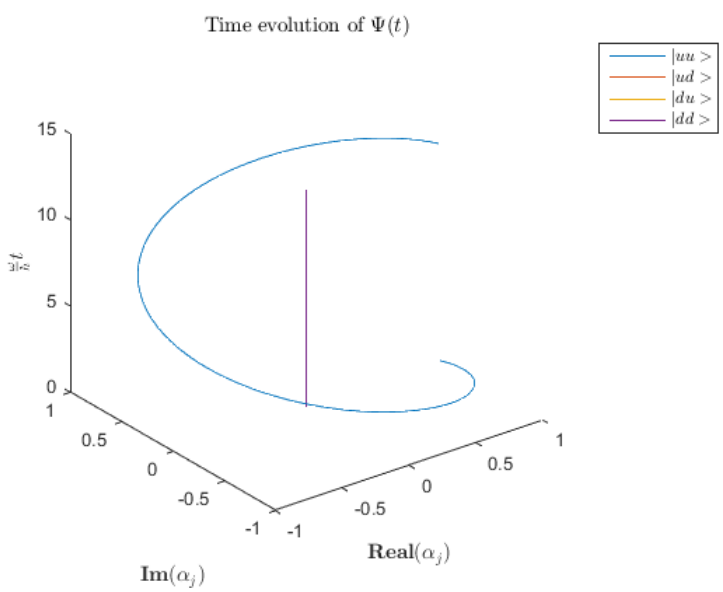
7. Now we substitute $\alpha_j(t) = \alpha_j(0)e^{-\frac{i}{\hbar}\lambda_j t}$

$$\begin{aligned}|\Psi(t)\rangle &= \frac{1}{\sqrt{2}}(e^{-\frac{i}{\hbar}\lambda_{T_2}t}\,|T_2\rangle + e^{-\frac{i}{\hbar}\lambda_{T_3}t}\,|T_3\rangle) \\ &= \frac{1}{\sqrt{2}}(e^{-\frac{i\omega}{2\hbar}t}\,|T_2\rangle + e^{-\frac{i\omega}{2\hbar}t}\,|T_3\rangle)\end{aligned}$$

We can visualize the amplitudes in Matlab as follows

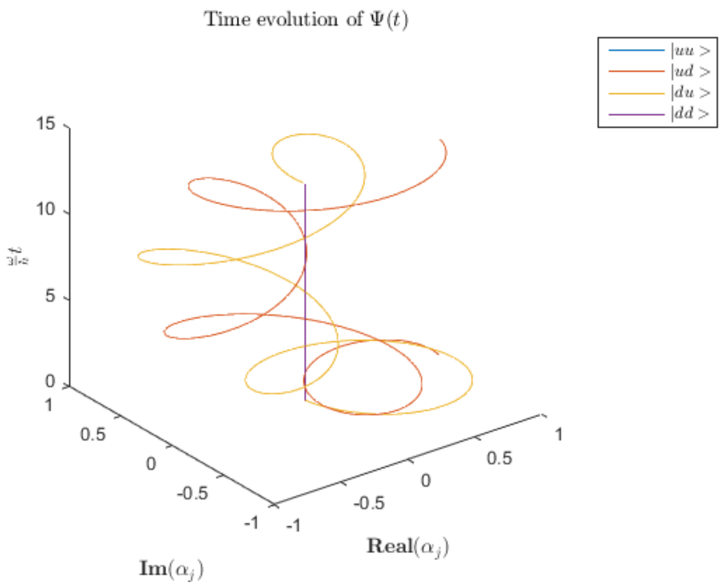
```
%% Time-dependence and entanglement

% 1-Spin state vectors to aid 2-spin constructions
u = [1;0];
d = [0;1];

% State vector
Psi = kron(u,u); %|uu>

% H's eigenvectors
sing = (1/sqrt(2))*(kron(u,d)-kron(d,u));
T_1 = (1/sqrt(2))*(kron(u,d)+kron(d,u));
T_2 = (1/sqrt(2))*(kron(u,u)+kron(d,d));
T_3 = (1/sqrt(2))*(kron(u,u)-kron(d,d));

% H's eigenvalues (times [omega])
lambda_s = -3/2;
lambda_t1 = 1/2;
lambda_t2 = 1/2;
lambda_t3 = 1/2;

% Computing alphas (t=0)
alpha_s = Psi'*sing; % ex. <Psi||sing>
alpha_t1 = Psi'*T_1;
alpha_t2 = Psi'*T_2;
```

```
alpha_t3 = Psi'*T_3;

% Time-dependence equation [tn = t*(omega/hbar)]
Psi_td = @(tn)(alpha_s*exp(-1i*lambda_s*tn)*sing+...
alpha_t1*exp(-1i*lambda_t1*tn)*T_1+...
alpha_t2*exp(-1i*lambda_t2*tn)*T_2+...
alpha_t3*exp(-1i*lambda_t3*tn)*T_3);

% Time period of plot
tn = 0:.05:1/((1/2)/(2*pi));
amplitudes = zeros(4,length(tn));
for j = 1:length(tn)
amplitudes(:,j) = Psi_td(tn(j));
end

% Plot amplitude in the complex plane and time as 3D
plot3(real(amplitudes(:,:))',imag(amplitudes(:,:))',tn')
legend({'$|uu>$','$|ud>$','$|du>$','$|dd>$'},...
'Interpreter','latex')
title('Time evolution of $\Psi(t)$','Interpreter','latex')
xlabel('$\textbf{Real}(\alpha_j)$','Interpreter','latex')
ylabel('$\textbf{Im}(\alpha_j)$','Interpreter','latex')
zlabel('$\frac{\omega}{\hbar}t$','Interpreter','latex',...
'fontsize',18)
```

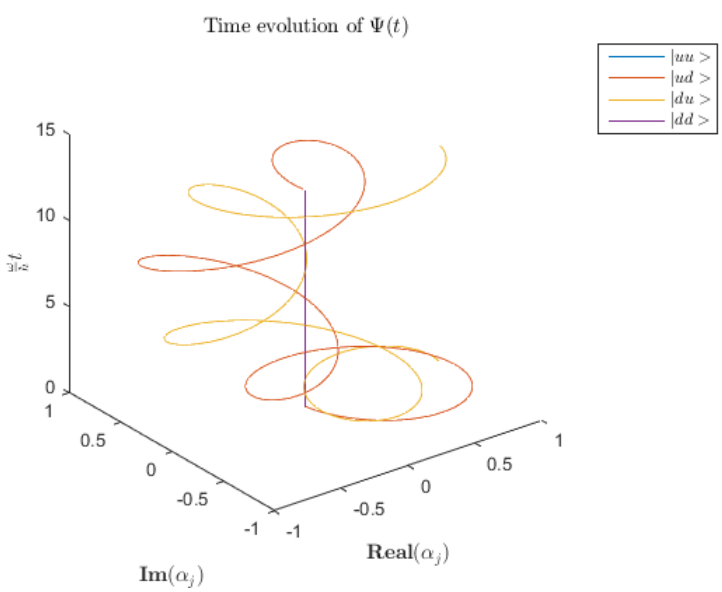



Now we can modify the previous code to produce similar results for:

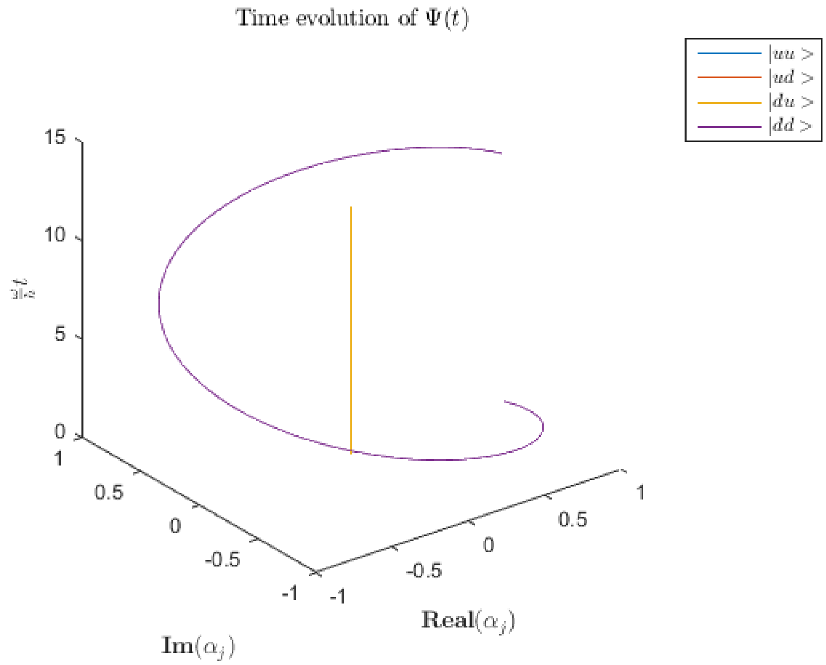
$\Psi(0) = |ud\rangle$




$\Psi(0) = |du\rangle$



$$\Psi(0) = |dd\rangle$$




**Note** that in these plots we are adding the eigenvectors and plotting the amplitudes $\alpha_j$ for the resulting base vectors $|uu\rangle$, $|ud\rangle$, $|du\rangle$, and $|dd\rangle$. For example

$$\frac{1}{\sqrt{2}}(|T_2\rangle + |T_3\rangle) = |uu\rangle$$

Which we plot as $\alpha_{uu} = 1$ and $\alpha_{ud} = \alpha_{du} = \alpha_{dd} = 0$.