Lecture Five Exercise Solutions

Exercise 5.1

Any 2 X 2 Hermitian matrix **L** can be written as a sum of four terms,

$$L = a\sigma_x + b\sigma_y + c\sigma_z + dI$$

Verify this claim.

Solution

Any 2X2 Hermitian matrix has the form

$$\begin{pmatrix} r & w \\ w^* & r' \end{pmatrix}$$

where the diagonal elements (r) are real and the other two are complex conjugates.

$$L = a\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + c\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + d\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} r & w \\ w^* & r' \end{pmatrix} = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} + \begin{pmatrix} 0 & -bi \\ bi & 0 \end{pmatrix} + \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} + \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}$$

$$\begin{pmatrix} r & w \\ w^* & r' \end{pmatrix} = \begin{pmatrix} d+c & a-bi \\ a+bi & d-c \end{pmatrix}$$

We can allow c and d to be real numbers specifying any arbitrary real numbers r and r', and a and b to be real numbers that specify any complex number w and its complex conjugate w^*.

Exercise 5.2

1) Show that $\Delta A^2 = \langle \bar{A}^2 \rangle$ and $\Delta B^2 = \langle \bar{B}^2 \rangle$.

2) Show that $[\bar{A}, \bar{B}] = $ [A , B].

3) Using these relations, show that

$$\Delta A \Delta B \geq \frac{1}{2}|\langle\Psi|[A,B]|\Psi\rangle|$$

Solution

1) Section 5.4 shows us how $\Delta A^2 = \langle\bar{A}^2\rangle$. We define the operator $\bar{A}$ to be

$$\bar{A} = A - \langle A\rangle$$

with eigenvalues

$$\bar{a} = a - \langle A\rangle$$

The standard deviation of A is defined by

$$\begin{aligned}(\Delta A)^2 &= \sum_a \bar{a}^2 P(a)\\ &= \sum_a (a - \langle A\rangle)^2 P(a)\\ &= \langle\Psi|\bar{A}^2|\Psi\rangle\\ &= \langle\bar{A}^2\rangle\end{aligned}$$

We can show $\Delta B^2 = \langle\bar{B}^2\rangle$ with the same approach.

2) We define $\bar{A} = A - \langle A\rangle$ and $\bar{B} = B - \langle B\rangle$ and substitute these into the given equality.

$$\begin{aligned}[\bar{A},\bar{B}] &= [(A - \langle A\rangle),(B - \langle B\rangle)]\\ &= (A - \langle A\rangle)(B - \langle B\rangle) - (B - \langle B\rangle)(A - \langle A\rangle)\\ &= AB - \langle A\rangle B - A\langle B\rangle + \langle A\rangle\langle B\rangle - BA + \langle B\rangle A + B\langle A\rangle - \langle B\rangle\langle A\rangle\\ &= AB - BA\\ &= [A,B]\end{aligned}$$

3) Replacing A and B with $\bar{A}$ and $\bar{B}$ in Equation (5.12), we have

$$\begin{aligned}2\sqrt{\langle\bar{A}^2\rangle\langle\bar{B}^2\rangle} &\geq |\langle\Psi|[\bar{A},\bar{B}]|\Psi\rangle|\\ 2\sqrt{\Delta A^2 \Delta B^2} &\geq |\langle\Psi|[A,B]|\Psi\rangle|\\ \Delta A^2 \Delta B^2 &\geq \frac{1}{2}|\langle\Psi|[A,B]|\Psi\rangle|\end{aligned}$$