Lecture Four Exercise Solutions

Exercise 4.1

Prove that if U is unitary, and if $|A\rangle$ and $|B\rangle$ are any two state-vectors, then the inner product of U$|A\rangle$ and U$|B\rangle$ is the same as the inner product of $|A\rangle$ and $|B\rangle$.

Solution

A unitary operator is one that satisfies the relationship $U^\dagger U = I$. Applying the inner product of state-vectors $|A\rangle$ and $|B\rangle$, we have

$$\langle A|\, B\rangle = \begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3\end{pmatrix} \begin{pmatrix}\beta_1 \\ \beta_2 \\ \beta_3\end{pmatrix} = \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3$$

$\Big\langle A\Big|\, U^\dagger U|B\Big\rangle = \langle A|\, I|B\rangle = \langle A|\, B\rangle$ from the definition provided above.

Exercise 4.2

Prove that if M and L are both Hermitian, i[M,L] is also Hermitian. Note that the i is important. The commutator is, by itself, not Hermitian.

Solution

From the definition of Hermitian, we have $M = M^\dagger = (M^T)^*$ and $L = L^\dagger = (L^T)^*$. To prove $i[M, L]$ is Hermitian, we must show $i[M, L] = (i[M, L])^\dagger$.

$$\begin{aligned}
(i[M,L])^\dagger &= (i[ML - LM])^\dagger \\
&= [(i[ML - LM])^T]^* \\
&= [(iML - iLM)^T]^* \\
&= [(iML)^T - (iLM)^T]^* \\
&= (i(L^T M^T))^* - (i(M^T L^T))^*] \\
&= -i((L^T)^*(M^T)^*) + i((M^T)^*(L^T)^*) \\
&= i(M^\dagger)(L^\dagger) - i(L^\dagger)(M^\dagger) \\
&= i[(M^\dagger)(L^\dagger) - (L^\dagger)(M^\dagger)] \\
&= i[ML - LM] \\
&= i[M, L]
\end{aligned}$$

Exercise 4.3

Go back to the definition of Poisson brackets in Volume 1 and check that the identification in Eq. 4.21 is dimensionally consistent. Show that without factor $\hbar$, it would not be.

Solution

Equation (4.21) gives us the formal identification between commutators and Poisson brackets: [F,G] $\Longleftrightarrow i\hbar\{F,G\}$

$$\dot{F} = \frac{\partial}{\partial t}(F) = \{F,G\} = \sum\left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial q_i}\frac{\partial F}{\partial p_i}\right)$$

where q(space) is measured in m and p (momentum) is measured in kg·m/s.

Rewriting this relationship solely in terms of units, we have

$$\{F,G\} = \left(\frac{F}{m}\frac{G}{kg\cdot m/s} - \frac{G}{m}\frac{F}{kg\cdot m/s}\right) = \left(\frac{F}{m}\frac{Gs}{kg\cdot m} - \frac{G}{m}\frac{Fs}{kg\cdot m}\right)$$

The units for $\hbar = \dfrac{h}{2\pi}$ are $kg\cdot m^2/s$ so that

$$i\hbar\{F,G\} = \frac{kg\cdot m^2}{s}\left(\frac{F}{m}\frac{Gs}{kg\cdot m} - \frac{G}{m}\frac{Fs}{kg\cdot m}\right) = FG{-}GF = [F,G] \text{ as desired.}$$

Exercise 4.4

Verify the commutation relations of Eqs 4.26.

$$[\sigma_x, \sigma_y] = 2i\sigma_z$$

$$[\sigma_y, \sigma_z] = 2i\sigma_x$$

$$[\sigma_z, \sigma_x] = 2i\sigma_y$$

Solution

$$
\begin{aligned}
[\sigma_x, \sigma_y] = \sigma_x\sigma_y - \sigma_y\sigma_x &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \\
&= \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} \\
&= 2i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= 2i\sigma_z
\end{aligned}
$$

$$
\begin{aligned}
[\sigma_y, \sigma_z] = \sigma_y\sigma_z - \sigma_z\sigma_y &= \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix} \\
&= 2i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= 2i\sigma_x
\end{aligned}
$$

$$
\begin{aligned}
[\sigma_z, \sigma_x] = \sigma_z\sigma_x - \sigma_x\sigma_z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \\
&= 2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
&= 2 \begin{pmatrix} 0 & -i^2 \\ i^2 & 0 \end{pmatrix} \\
&= 2i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
&= 2i\sigma_y
\end{aligned}
$$

Exercise 4.5

Take any unit 3-vector $\vec{n}$ and form the operator $H = \frac{\hbar\omega}{2}\sigma\cdot\vec{n}$. Find the energy eigenvalues and eigenvectors by solving the time-independent Schrodinger equation. Recall that Eq. 3.23 gives $\sigma_n \cdot \vec{n}$ in component form.

Solution

Recall that in Exercise 3.4, we solved for the eigenfunctions and eigenvalues of σ_n, defined as

$$\sigma_n = \begin{pmatrix} n_z & (n_x - in_y) \\ (n_x + in_y) & -n_z \end{pmatrix}$$

whose eigenvalues and eigenvectors are $\lambda_1 = 1$ with $|\lambda_1\rangle = \begin{pmatrix} cos\frac{\theta}{2} \\ e^{i\phi}sin\frac{\theta}{2} \end{pmatrix}$ and $\lambda_2 = -1$ with $|\lambda_2\rangle \begin{pmatrix} sin\frac{\theta}{2} \\ e^{-i\phi}cos\frac{\theta}{2} \end{pmatrix}$.

The given Hamiltonian can be rewritten as $H = \frac{\hbar\omega}{2}\sigma_n$, so that the eigenvalues of H are $\pm\frac{\hbar\omega}{2}$ and the eigenvectors of H remain the same as those of σ_n.

Exercise 4.6

Carry out the Schrodinger Ket recipe for a single spin. The Hamiltonian is $H = \frac{\hbar\omega}{2}\sigma_z$ and the final observable is σ_x. The initial state is given as $|u\rangle$. After time t, an experiment is done to measure σ_y. What are the possible outcomes and what are the probabilities for those outcomes?

Solution

Recipe for a Schrodinger Ket:

1. The Hamiltonian is given as $H = \frac{\hbar\omega}{2}\sigma_z$.

2. The initial state $|\psi(0)\rangle$ is $1|u\rangle + 0\ |d\rangle = |u\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

3. The eigenvalues and eigenvectors of H differ from those of σ_z by a real constant $\frac{\hbar\omega}{2}$ so that we have

$$E_1 = \frac{\hbar\omega}{2} \Rightarrow |E_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$E_2 = \frac{\hbar\omega}{2} \Rightarrow |E_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

4. The initial coefficients $a_j(0)$ of $|\psi(0)\rangle$ are

$$a_1(0) = \langle E_1|\ \psi(0)\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$$

$$a_2(0) = \langle E_2|\ \psi(0)\rangle = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$$

5. Rewriting in terms of the eigenvectors of H

$$|\psi(0)\rangle = \sum_j a_j(0)\,|E_j\rangle = 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

6, 7. Rewrite the state vector $|\psi(t)\rangle$ in terms of the initial coefficients.

$$\begin{aligned} |\psi(t)\rangle &= \sum_j a_j(t)\,|E_j\rangle \\ &= \sum_j a_j(0) e^{-i/hE_j t}\,|E_j\rangle \\ &= a_1(0) e^{-i/hE_1 t}\,|E_1\rangle + a_2(0) e^{-i/hE_2 t}\,|E_2\rangle \\ &= 1 e^{-iwt/2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} e^{-iwt/2} \\ 0 \end{pmatrix} \end{aligned}$$

Now that we have $|\psi(t)\rangle$, we can predict the probabilities for each possible outcome of an experiment as a function of time. Measuring σ_y gives us

$$
\begin{aligned}
P_{+1}(t) &= \langle\lambda_1|\ \psi(t)\rangle\,\langle\psi(t)|\ \lambda_1\rangle \\
&= \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}e^{-iwt/2}\\ 0\end{pmatrix}\begin{pmatrix}e^{iwt/2} & 0\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{i}{\sqrt{2}}\end{pmatrix} \\
&= (\frac{1}{\sqrt{2}}e^{-iwt/2})(\frac{1}{\sqrt{2}}e^{iwt/2}) \\
&= \frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
P_{-1}(t) &= \langle\lambda_2|\ \psi(t)\rangle\,\langle\psi(t)|\ \lambda_2\rangle \\
&= \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}e^{-iwt/2}\\ 0\end{pmatrix}\begin{pmatrix}e^{iwt/2} & 0\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{-i}{\sqrt{2}}\end{pmatrix} \\
&= (\frac{1}{\sqrt{2}}e^{-iwt/2})(\frac{1}{\sqrt{2}}e^{iwt/2}) \\
&= \frac{1}{2}
\end{aligned}
$$