Lecture Three Exercise Solutions

Exercise 3.1

Prove the following: If a vector space is N-dimensional, an orthonormal basis of N vectors can be constructed from the eigenvectors of a Hermitian operator.

Solution

Let V^N be an N dimensional vector space and L be a Hermitian operator in V^N. Then L can be represented by an N x N matrix and we can solve the basic eigenvector equation for a non-zero vector $|\lambda\rangle$ such that L$|\lambda\rangle = \lambda\,|\lambda\rangle$. We know $det(A - \lambda I) = 0$ is an n^{th} degree polynomial, which can be solved for N eigenvalues and N corresponding eigenvectors. These eigenvectors will be orthogonal (or can be chosen to be orthogonal) and form a complete set (from the fundamental theorem). This means in an N dimensional vector space, there can always be found N mutually orthogonal eigenvectors of an NxN Hermitian operator, which can be normalized to form an orthonormal basis of V^N.

Exercise 3.2

Prove that Eq. 3.16 is the unique solution to Eqs. 3.14 and 3.15.

Solution

$$\begin{pmatrix} (\sigma_z)_{11} & (\sigma_z)_{12} \\ (\sigma_z)_{21} & (\sigma_z)_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{c} (\sigma_z)_{11} = 1 \\ (\sigma_z)_{21} = 0 \end{array}$$

$$\begin{pmatrix} (\sigma_z)_{11} & (\sigma_z)_{12} \\ (\sigma_z)_{21} & (\sigma_z)_{22} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \begin{array}{c} (\sigma_z)_{12} = 0 \\ (\sigma_z)_{22} = -1 \end{array}$$

$$\begin{pmatrix} (\sigma_z)_{11} & (\sigma_z)_{12} \\ (\sigma_z)_{21} & (\sigma_z)_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We know this solution is unique because the determinant of matrix (3.16) is non-zero.

Exercise 3.3

Calculate the eigenvectors and eigenvalues of σ_n.

Solution

We are asked to solve the eigenvector equation of the form $\sigma_n |\lambda\rangle = \lambda |\lambda\rangle$

$\sigma_n |\lambda\rangle = \lambda |\lambda\rangle \rightarrow \begin{pmatrix} cos\theta & sin\theta \\ sin\theta & -cos\theta \end{pmatrix} \begin{pmatrix} cos\alpha \\ sin\alpha \end{pmatrix} = \lambda \begin{pmatrix} cos\alpha \\ sin\alpha \end{pmatrix}$

Expanding these products, we have

$$cos\theta\ cos\alpha + sin\theta\ sin\alpha = \lambda\ cos\alpha \Rightarrow cos(\theta - \alpha) = \lambda\ cos\alpha$$
$$sin\theta\ cos\alpha - cos\theta\ sin\alpha = \lambda\ sin\alpha \Rightarrow sin(\theta - \alpha) = \lambda\ sin\alpha$$

Solving both equations in terms of λ :

$$\lambda = \frac{cos(\theta - \alpha)}{cos\alpha}$$

$$\lambda = \frac{sin(\theta - \alpha)}{sin\alpha}$$

$$\frac{cos(\theta - \alpha)}{cos\alpha} = \frac{sin(\theta - \alpha)}{sin\alpha}$$

$$\begin{aligned} sin\alpha\ cos(\theta - \alpha) &= cos\alpha\ sin(\theta - \alpha) \\ sin\alpha\ cos(\theta - \alpha) - cos\alpha\ sin(\theta - \alpha) &= 0 \\ sin(\alpha - (\theta - \alpha)) &= 0 \\ sin(2\alpha - \theta) &= 0 \end{aligned}$$

Solving for α in terms of θ gives us $\alpha = \theta/2$ or $\alpha = \theta/2 + \pi/2$ and the following eigenvalues.

$$\lambda_1 = cos(\theta - \frac{\theta}{2})/cos\frac{\theta}{2} = 1$$

$$\lambda_2 = cos(\theta - \theta - \frac{\theta}{2})/cos\frac{\pi}{2} + \frac{\theta}{2} = -1$$

For $\lambda_1 = 1, |\lambda_1\rangle = \begin{pmatrix} cos\alpha_1 \\ sin\alpha_1 \end{pmatrix} = \begin{pmatrix} cos\frac{\theta}{2} \\ sin\frac{\theta}{2} \end{pmatrix}$

For $\lambda_2 = -1, |\lambda_2\rangle = \begin{pmatrix} cos\alpha_2 \\ sin\alpha_2 \end{pmatrix} = \begin{pmatrix} cos\frac{\pi}{2} + \frac{\theta}{2} \\ sin\frac{\pi}{2} + \frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} -sin(\frac{\theta}{2}) \\ cos(\frac{\theta}{2}) \end{pmatrix}$

Exercise 3.4

Let $n_z = cos\theta, n_x = sin\theta\ cos\phi$ and $n_y = sin\theta\ sin\phi$. Compute the eigenvalues and eigenvectors for the matrix of Eq. 3.23.

Solution

$$\sigma_n = \begin{pmatrix} n_z & (n_x - in_y) \\ (n_x + in_y) & -n_z \end{pmatrix} = \begin{pmatrix} cos\theta & sin\theta cos\phi - isin\theta sin\phi \\ sin\theta cos\phi + isin\theta sin\phi & -cos\theta \end{pmatrix}$$

Notice we can rewrite the entries of σ_n in a more efficient form.

$$\sigma_n = \begin{pmatrix} cos\theta & sin\theta e^{-i\phi} \\ sin\theta e^{i\phi} & -cos\theta \end{pmatrix}$$

We can assume the eigenvectors are of similar form as those of Exercise 3.3 (with an arbitrary phase change) and we are solving the eigenvector equation

$$\sigma_n |\lambda\rangle = \lambda |\lambda\rangle \rightarrow \begin{pmatrix} cos\theta & e^{-i\phi}sin\theta \\ e^{i\phi}sin\theta & -cos\theta \end{pmatrix} \begin{pmatrix} cos\alpha \\ sin\alpha \end{pmatrix} = \lambda \begin{pmatrix} cos\alpha \\ sin\alpha \end{pmatrix}$$

$$cos\theta\ cos\alpha + e^{-i\phi}sin\theta\ sin\alpha = \lambda\ cos\alpha$$
$$e^{i\phi}sin\theta\ cos\alpha - cos\theta\ sin\alpha = \lambda\ sin\alpha$$

We can solve for $e^{i\phi}$ in both equations.

$$e^{i\phi} = \frac{sin\theta sin\alpha}{\lambda cos\alpha - cos\theta cos\alpha}$$
$$e^{i\phi} = \frac{\lambda sin\alpha + cos\theta sin\alpha}{sin\theta cos\alpha}$$

$$\frac{sin\theta sin\alpha}{\lambda cos\alpha - cos\theta cos\alpha} = \frac{\lambda sin\alpha + cos\theta sin\alpha}{sin\theta cos\alpha}$$

Solving for λ gives us

$$sin^{\theta} + cos^2\theta = \lambda^2 \Rightarrow \lambda = \pm 1$$

To find the eigenvectors of the matrix, we must solve the following for the correct coefficients.

$$\sigma_n |\lambda\rangle = \lambda |\lambda\rangle \rightarrow \begin{pmatrix} cos\theta & e^{-i\phi}sin\theta \\ e^{i\phi}sin\theta & -cos\theta \end{pmatrix} \begin{pmatrix} a\ cos\frac{\theta}{2} \\ b\ sin\frac{\theta}{2} \end{pmatrix} = 1 \begin{pmatrix} a\ cos\frac{\theta}{2} \\ b\ sin\frac{\theta}{2} \end{pmatrix}$$

$$a\ cos\theta cos\frac{\theta}{2} + b\ e^{-i\phi}sin\theta\ sin\frac{\theta}{2} = a\ cos\frac{\theta}{2}$$
$$a\ e^{i\phi}sin\theta\ cos\frac{\theta}{2} - b\ cos\theta\ sin\frac{\theta}{2} = b\ sin\frac{\theta}{2}$$

After dividing the top equation by a, we have

$$cos\theta\ cos\frac{\theta}{2} + \frac{b}{a}e^{-i\phi}sin\theta\ sin\frac{\theta}{2} = cos\frac{\theta}{2}$$

Let $b = e^{i\phi}$ and $a = 1$ so that the $e^{-i\phi}$ term cancels accordingly. We can simplify again using the additive trig identity to obtain the equations from Exercise 3.3 and see that for $\lambda_1 = 1$,

$$|\lambda_1\rangle = \begin{pmatrix} acos\frac{\theta}{2} \\ bsin\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} 1cos\frac{\theta}{2} \\ e^{i\phi}sin\frac{\theta}{2} \end{pmatrix}$$

Similarly, for $\lambda_2 = -1$, we can apply a similar form of reasoning and find coefficients to the eigenvector from exercise 3.3 so that $|\lambda_2\rangle$ is orthogonal to $|\lambda_1\rangle$.

$$\begin{pmatrix} cos\frac{\theta}{2} & e^{i\phi}sin\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} c\ sin\frac{\theta}{2} \\ -d\ cos\frac{\theta}{2} \end{pmatrix} = 0$$

$$cos\frac{\theta}{2}\ csin\frac{\theta}{2} - e^{i\phi}sin\frac{\theta}{2}\ dcos\frac{\theta}{2} = 0$$

Notice that c = 1 and d = $e^{-i\phi}$ are solutions to the equations.

For $\lambda_2 = -1, |\lambda_2\rangle = \begin{pmatrix} csin\frac{\theta}{2} \\ -dcos\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} 1sin\frac{\theta}{2} \\ e^{-i\phi}cos\frac{\theta}{2} \end{pmatrix}$

Exercise 3.5

Suppose that a spin is prepared so that $\sigma_m = +1$. The apparatus is then rotated to the $\hat{n}$ direction and σ_n is measured. What is the probability that the result is +1?

Solution

From exercise 3.4, we know the eigenvector of an arbitrary n axis is given by

$$|+\lambda\rangle = \begin{pmatrix} 1cos\frac{\theta}{2} \\ e^{i\phi}sin\frac{\theta}{2} \end{pmatrix}$$

$$P(+n) = || < +m| + n > ||^2 = || \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1cos\frac{\theta}{2} \\ e^{i\phi}sin\frac{\theta}{2} \end{pmatrix} ||^2 = cos^2\frac{\theta}{2}$$